Comment on "Linear instability of magnetic Taylor-Couette flow with Hall effect"

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In the paper we comment on [Rüdiger and Shalybkov (RS), Phys. Rev. E **69**, 016303 (2004)], the instability of the Taylor-Couette flow interacting with a homogeneous background field subject to the Hall effect is studied. We correct a falsely generalizing interpretation of results presented there which could be taken to disprove the existence of the Hall-drift-induced magnetic instability described in Rheinhardt and Geppert, Phys. Rev. Lett. **88**, 101103 (2002). It is shown that, in contrast to what is suggested by RS, no additional shear flow is necessary to enable such an instability with a nonpotential magnetic background field, whereas for a curl-free one it is. In the latter case, the instabilities found in RS in situations where neither a hydrodynamic nor a magnetorotational instability exists are demonstrated to be most likely magnetic instead of magnetohydrodynamic. Further, some minor inaccuracies are clarified.

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I. NECESSARY CONDITIONS FOR FLOW INSTABILITIES WITH HALL EFFECT

The main purpose of this Comment on the paper in Ref. [1] (hereafter referred to as RS) is to prevent an incorrect conclusion with respect to our work [2,3] which could be drawn from an incorrect statement in the discussion section of RS. There, at the end of the third paragraph, the authors conclude from the invariance of their results with respect to simultaneous sign inversions of shear and Hall term that no instabilities are possible without shear. Although this conclusion, being looked at out of context, is not comprehensible, it is nevertheless true for the special case of a homogeneous (more generally, curl-free) background field $\boldsymbol{B}_0$, but not in general. As the scheme (40) of RS is valid for nonpotential (axisymmetric) fields, too, and the quoted conclusion is drawn completely on its basis, the reader will be tempted to generalize it. He or she could then come to the conclusion that the results on a Hall instability *without* shear reported in [2,3] have to be called into question. (Note that the term "shear" is used throughout RS to refer to the macroscopic motion of a fluid.) Here, we will show that conclusions on necessary conditions for the instabilities in question can reliably be drawn on the basis of energy considerations. They support the possibility of a Hall instability without shear.

The linearized induction and Navier-Stokes equations describing the evolution of small perturbations $\boldsymbol{B}'$ and $\boldsymbol{u}'$ of the background field $\boldsymbol{B}_0$ and the shear flow (here, differential rotation) $\boldsymbol{u}_0$, respectively, read for a curl-free $\boldsymbol{B}_0$

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \mathrm{curl}(\boldsymbol{u}_0 \times \boldsymbol{B}' + \boldsymbol{u}' \times \boldsymbol{B}_0) + \eta \Delta \boldsymbol{B}' - \beta\, \mathrm{curl}(\mathrm{curl}\, \boldsymbol{B}' \times \boldsymbol{B}_0), \quad (1)$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + (\boldsymbol{u}'\,\nabla)\boldsymbol{u}_0 + (\boldsymbol{u}_0\,\nabla)\boldsymbol{u}' = -\nabla p'/\rho + \nu \Delta \boldsymbol{u}' + \mathrm{curl}\, \boldsymbol{B}' \times \boldsymbol{B}_0/(\mu_0 \rho), \quad (2)$$

where we used the symbols introduced in RS. Standard arguments yield the following evolution equation for the total energy E of the perturbations:

$$\frac{dE}{dt} = \frac{1}{2}\frac{d}{dt}\left(\int_{V'} \boldsymbol{B}'^2/\mu_0\, dV + \int_V \rho \boldsymbol{u}'^2 dV\right) = -\int_V \left[(\mathrm{curl}\, \boldsymbol{B}')^2/(\mu_0^2\sigma) + \rho\nu(\mathrm{curl}\, \boldsymbol{u}')^2\right] dV + \int_V \mathrm{curl}\, \boldsymbol{B}' \cdot (\boldsymbol{u}_0 \times \boldsymbol{B}')dV/\mu_0 - \rho \int_V \mathrm{curl}\, \boldsymbol{u}' \cdot (\boldsymbol{u}_0 \times \boldsymbol{u}')dV \quad (3)$$

with V' being the infinite space minus any volume with infinite conductivity and V the volume of the container. Of course, solutions with growing total energy are impossible, as long as $\boldsymbol{u}_0 = \boldsymbol{0}$. More generally, even if we were to admit a rigid body motion for $\boldsymbol{u}_0$, growing solutions do not exist.

The situation changes qualitatively, if $\boldsymbol{B}_0$ is no longer curl-free: The additional term $-\beta\, \mathrm{curl}(\mathrm{curl}\, \boldsymbol{B}_0 \times \boldsymbol{B}')$ occurring in the linearized induction equation results in the additional energy term

$$-\beta \int_V \mathrm{curl}\, \boldsymbol{B}' \cdot (\mathrm{curl}\, \boldsymbol{B}_0 \times \boldsymbol{B}')dV/\mu_0, \quad (4)$$

which quite analogously to the term $\int_V \mathrm{curl}\, \boldsymbol{B}' \cdot (\boldsymbol{u}_0 \times \boldsymbol{B}')dV/\mu_0$ is potentially capable of delivering energy. Hence, the argument concerning the necessity of shear for the occurrence of an instability in the model of RS in fact supports our findings in [2,3], when the term "shear" is no longer used to refer to macroscopic motions only, but is extended to the microscopic motions of the carriers creating the

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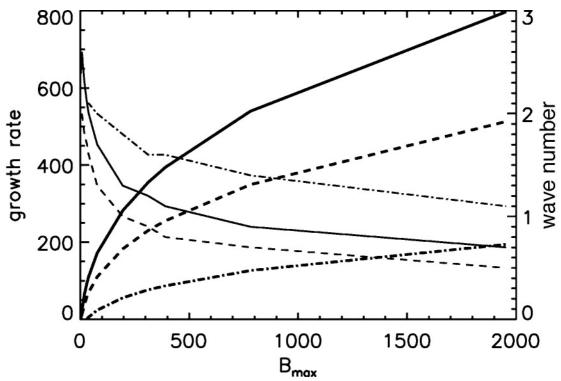

FIG. 1. Growth rates (thick) and wave numbers (thin) of the most unstable axisymmetric magnetic-field modes (kinematic case, $\boldsymbol{u}'=\mathbf{0}$) as a function of the background field strength. Length, time, and magnetic field are normalized by $R_{\rm out}$, $R_{\rm out}^2/\eta$, and η/β, respectively. $B_{\max}$ represents the maximum of the background field profile. $R_{\rm in}/R_{\rm out}=\hat{\eta}=0.5$. Boundary conditions as defined in RS. Vacuum-vacuum: solid; inner perfect conductor–outer vacuum: dashed; perfect conductor–perfect conductor: dot-dashed. The eigenmodes are nonoscillatory in the first two cases, but oscillatory in the third. Interestingly, for the vacuum-vacuum boundary condition, the instability emerges roughly at $B_{\max}\gtrsim 3$ as in the plane model.

current curl $\boldsymbol{B}_0/\mu_0$. If the latter should be capable of replacing the shear velocity $\boldsymbol{u}_0$, it must not be interpretable as a rigid body motion. Therefore, a background field exhibiting a sufficiently curved profile is a necessary condition for the occurrence of the instability we reported on, as we stressed in all our papers on this issue. (A suitable profile for a plane slab $-1\leqslant z\leqslant 1$ with its normal in the z direction is, for instance, $\boldsymbol{B}_0=\hat{B}_0(1-z^2)\boldsymbol{e}_x$, as used in [2].)

As the energy term (4) contains only magnetic fields, the possibility exists that even in the absence of any macroscopic motions ($\boldsymbol{u}'=\boldsymbol{u}_0=\mathbf{0}$), say in a crystallized neutron star crust, nevertheless an instability may occur. For plane geometry we demonstrated that this possibility is real both for a uniform [2] and a stratified slab [4]. In the cylindrical geometry considered in RS, the instability occurs as well. Figure 1 shows normalized growth rates and wave numbers of the most rapidly growing axisymmetric modes versus the normalized strength of the background field. Its profile was specified as $\boldsymbol{B}_0(R)\propto(R-R_{\rm in})^2(R-R_{\rm out})^2\boldsymbol{e}_z$.

The second additional energy term due to a nonpotential background field,

$$\int_V \boldsymbol{u}'\cdot(\mathrm{curl}\,\boldsymbol{B}_0\times\boldsymbol{B}')dV/\mu_0, \qquad (5)$$

is capable of delivering or consuming energy, too. Corresponding instabilities exist and are (for $\boldsymbol{u}_0=\mathbf{0}$) usually referred to as unstable Alfvén modes (see, e.g., [5]). Their nature is obviously MHD, as Eq. (5) vanishes for $\boldsymbol{u}'=\mathbf{0}$ and/or $\boldsymbol{B}'=\mathbf{0}$.

II. MAGNETIC VS MHD NATURE OF INSTABILITIES

Another remark connected with the above seems to be appropriate. In the Results and Discussion sections of RS, the impression is given that the reported instability is primarily one of the flow. In our opinion, there are good reasons, and moreover even evidence provided by RS itself, to interpret this instability as a primarily magnetic (and not MHD) one for conditions in which the flow would be stable otherwise (that is, for "positive shear" and for "negative shear" with subcritical Reynolds numbers, i.e., for the part of the Re-Ha plane beneath the dashed line in Fig. 6 of RS).

Considering the induction equation including differential rotation and Hall effect with the velocity perturbations suppressed (i.e., the kinematic case), one has formally the same equation as that which describes mean-field dynamos due to differential rotation and the so-called $\boldsymbol{\omega}\times\boldsymbol{j}$ effect; see [6,7]. From these calculations and also from qualitative considerations, it follows that the sign relation between Hartmann number Ha and $d\Omega/dR$ reported in RS (see, e.g., Sec. III) is just the one necessary for dynamo action, that is, a *magnetic* instability. (Note that Cowling's theorem does not apply.) In Sec. III B of RS, a marginal curve in the Re-Ha plane is given for the kinematic case $\boldsymbol{u}'=\mathbf{0}$. In that part of the plane where no hydrodynamic or magnetorotational instability (MRI) exists, it practically coincides with the marginal curve of the full system's instability. Thus, one may suppose that the velocity perturbations are simply "enslaved" by the magnetic ones in cases in which no instabilities occur without the Hall effect. Since an enslaved $\boldsymbol{u}'$ gives rise to additional dissipation, the full system should exhibit smaller growth rates compared with those of the kinematic case. A hint on this is provided by Fig. 6 of RS, showing that for $1\lesssim\mathrm{Ha}\lesssim 7$ in the full system a slightly stronger differential rotation is needed for marginal stability than in the kinematic case. To judge the nature of the instability, the signs of those integrals in Eq. (3) resulting from the potentially energy-delivering terms with the calculated eigensolutions inserted could be inspected.

When assuming the primarily magnetic character of the instability, its suppression with growing (absolute value of the) Hartmann number (cf. Figs. 2–4, 7, and 8 of RS) can be explained by the competition of two counteracting effects: On the one hand, growing $|\mathrm{Ha}|$ means growing dominance of the energy-delivering advection term, $\mathrm{curl}(\boldsymbol{u}_0\times\boldsymbol{B}')$, in Eq. (1) (by virtue of the admittedly energetically neutral but "catalyzing" Hall term) over the dissipation term. But on the other hand, it means also growing efficiency of the Lorentz force in Eq. (2) which causes growing dissipation due to the enslaved velocity perturbations which drain their energy by virtue of the second advection term, $\mathrm{curl}(\boldsymbol{u}'\times\boldsymbol{B}_0)$, in Eq. (1) out of the magnetic perturbations. Hence, the occurrence of a minimum with respect to Ha is quite natural.

III. CURRENT–FREE SOLUTION

Within the discussion of Fig. 5 of RS (Sec. III A), it is falsely stated that the existence of the current-free marginal solution $B_R'=B_z'=0$, $B_\phi'\propto R^{-1}$, $\boldsymbol{u}'=\mathbf{0}$ requires *both* boundary conditions to be those of the perfect conductor. In fact, there is no reason why such a current-free (or vacuum) solution could not continue from the inner boundary $R=R_{\rm in}$ on to infinity, which means nothing more than satisfying the corresponding vacuum condition at the outer rim $R=R_{\mathrm{out}}$. The necessary and sufficient condition for the existence of this vacuum solution everywhere outside the surface $R=R_{\mathrm{in}}$ is the existence of a net current in the z direction enclosed by this surface. Because an outer electromotive force is missing, a perfect conductor in the interior of the inner cylinder is needed. Then, e.g., an arbitrary surface current can flow without losses and therefore endlessly.

However, the dashed and the dot-dashed curves in Fig. 5 which correspond to the perfectly conducting inner cylinder are incomprehensible anyway, as they both should coincide with the curve $k=0$: This figure is intended to show the wave number belonging to the marginal eigensolution with the minimum Reynolds number for a given Hartmann number. In case the inner boundary condition is "perfect conductor," the vacuum solution always exists, which is marginal and allows also $\mathrm{Re}=0$ (or, equivalently, $\boldsymbol{u}_0=\mathbf{0}$), i.e., is associated with the minimum possible Re. This solution is characterized by $k=0$ and the mentioned curves should show that, except when there were other marginal solutions with $\mathrm{Re}=0$, but $k \neq 0$. But such solutions do not exist, because for $\mathrm{Re}=0$ and any Ha there is no (potentially) energy-delivering term in the induction equation [see Eq. (3)] and the only possibility for a marginal (i.e., nondecaying) solution is the vacuum one. Thus it appears that the vacuum solution was excluded for unknown reasons from the analysis which led to Fig. 5 and can therefore not be referred to to explain it.

However again, even then Fig. 5 is in disagreement with Fig. 2 of RS. The dashed line in the former should correspond to the solid line in the latter figure, which shows no special behavior at $\mathrm{Ha}=-2$ where k becomes zero in Fig. 5.

IV. MISSING REFERENCES

Considerations of the effect of turbulence on the Hall coefficient do exist [8,9] (see the last paragraph of RS). Perhaps it is appropriate to mention here a recent revival of mean-field Hall electrodynamics in [10], although there only the effect of the Hall drift onto the α coefficient is considered.

[1] G. Rüdiger and D. Shalybkov, Phys. Rev. E **69**, 016303 (2004).

[2] M. Rheinhardt and U. Geppert, Phys. Rev. Lett. **88**, 101103 (2002).

[3] U. Geppert and M. Rheinhardt, Astron. Astrophys. **392**, 1015 (2002).

[4] M. Rheinhardt, D. Konenkov, and U. Geppert, Astron. Astrophys. **420**, 631 (2004).

[5] R. Cross, *An Introduction to Alfvén Waves* (Adam Hilger, Bristol, 1988).

[6] K.-H. Rädler, Monatsber. Dtsch. Akad. Wiss. Berlin **11**, 194 (1969).

[7] H. K. Moffatt and M. R. E. Proctor, Geophys. Astrophys. Fluid Dyn. **21**, 265 (1982).

[8] G. Helmis, Monatsber. Dtsch. Akad. Wiss. Berlin **10**, 280 (1968).

[9] G. Helmis, Beitr. Plasmaphys. **11**, 417 (1971).

[10] P. Mininni, D. Gomez, and M. S. Mahajan, Astrophys. J. Lett. **567**, L81 (2002).